Chaotic Dynamics of a Third Order PLL with Resonant Low Pass Filter in Face of CW and FM Input Signals

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Abstract— **Nonlinear dynamics of a third order phase locked loop (PLL) using a resonant low pass filter in the face of continuous wave (CW) and frequency modulated (FM) input signals is examined. The role of design parameters of the loop resonant filter and the modulation index of the input FM signal on the system dynamics is studied numerically as well as experimentally. The occurrence of chaotic oscillations in the PLL is verified by evaluating some well-known chaos quantifiers like Lyapunov Exponents from the numerical time series data.**

Index Terms— **Third Order PLL, Resonant Filter, Stability Analysis, Chaotic dynamics, Lyapunov Exponent.**

I. Introduction

In recent years, chaotic oscillations in PLLs have been reported by several authors in the literature [1-4]. PLLs are used by communication system designers as both modulators and demodulators and chaotic oscillations in them could be a matter of both problem and benefit from the application point of view. The self generation of chaotic oscillations in third order PLLs and the inherent synchronization property of PLLs could be easily exploited in the design of coherent receivers of chaos-based communication systems. In the work of Endo et al., chaos from PLL based frequency modulators was reported in the condition when the carrier frequency of the FM signal was outside the locked state of the PLL. In this paper we report about the chaotic oscillations that are observed in an FM demodulator based on a third order PLL with the carrier signal within the lock range of the PLL. JRC Piqueira analyzed the stability and determined the lock in range of a similar type PLL form the view point of nonlinear dynamics in [5]. The present work shows that the PLL breaks into chaotic oscillations with the increase of the gain of the loop filter beyond a limit with CW input signal. The gain of the filter and its time constant can be the control design parameters of the PLL system to generate the chaos. Chaos is also observed with an FM input signal, the modulation index of the FM signal can be an additional control parameter of the generated chaos for both in tune and off tune carrier conditions. The proposed system can provide both baseband chaos and chaos-modulated RF signal obtainable at the control input terminal and the output terminal of the loop voltage-control oscillator (VCO) respectively.

We organize the paper in the following way. In section-II, the mathematical model of the system is formulated with its stability analysis. Section-III describes the results of numerical simulation study by solving the system equations using fourth order Runge-Kutta (RK) method. Chaotic dynamics of the system is examined by phase-plane portraits of the state-variables as well as by calculating the Lyapunov Exponents for different values of design parameters. The details of experimental studies on a prototype hardware system are given in section-IV. In the last section, some discussions on the importance of the study are presented.

II. Formulation Of System Equations And Stability Condition

Phase locked loop consists of three basic components: phase detector (PD), Low pass filter and Voltage control oscillator (VCO) as depicted in Fig. 1. Schematic diagram of the system is given in Fig. 2. A multiplier type phase detector and a resonant low pass filter are used in the loop. A FM signal of the form

$$s_{in}(t) = \sin\{\omega_c t - \psi(t)\} \tag{1}$$

is applied to the input of the PD. Where ω_c is the carrier frequency and the phase variable $\psi(t)$ denotes the frequency modulation. For single tone modulation

$$\psi = m\cos\omega_m t \tag{2}$$

Where, ω_m is the frequency of the modulating signal with *m* modulating index. To study the dynamics of PLL with CW input simply one can put ψ=0. Let the VCO output signal be denoted by $2\cos\{\omega_r t + \theta_r(t)\}$; Where ω_r the free running frequency of VCO and $\theta_r(t)$ is the phase.
The PD produces a output proportional to the sine of phase difference , i.e.,

$$\sin\{x(t) - \psi(t)\}$$

where, $x(t) = (\omega_c - \omega_r)t - \theta_r(t)$

This PD output is the input of the resonant filter circuit. We take two state variable y and z just before the feedback capacitor of the filter and at the output of the filter respectively. Using OP-AMP circuit equation, one can obtain the following two equations,

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$$\frac{dy}{dt} = a\sin(x-\Psi) + a(g-2)y - a\left(\frac{g-1}{g}\right)z \tag{3}$$

$$\frac{dz}{dt} = agy - az \tag{4}$$

Where g is the gain of the filter defined as $g = 1 + \frac{R_2}{R_1}$ and $a = \frac{1}{T} = \frac{1}{CR}$ where, C, R and T are the capacitance, resistance and time constant that are used in the filter circuit. VCO's oscillating frequency changes about the free running frequency (ω_r) in accordance with the controlling voltage z. From the basic PLL theory, one can write the phase part of the reference signal $\theta_r(t)$ as the time integrated version of the control input to the VCO (with sensitivity k). Thus,

$$\frac{d\theta_r}{dt} = kz \tag{5}$$

$$\frac{dx}{dt} = \Omega - kz \tag{6}$$

Where $\Omega = \omega_r - \omega_c$ is the frequency detunning and k is loop gain. For calculation one can define another variable Õ which define as,

$$\phi = \omega_m t \tag{7}$$

Equation (7) and (2) are first differentiated and then normalised whereas (3),(4),(6) are directly normalised with the factor $\tau = at$; a dimentinless quantity. Finally one get the following five equations(8a-8e) that describes the dynamics of the system with FM input.

$$\frac{d\varphi}{d\tau} = \frac{\omega_m}{a} = f_n \tag{8a}$$

$$\frac{d\Psi}{d\tau} = \frac{-m\omega_m}{a} sin\ (\varphi) \tag{8b}$$

$$\frac{dx}{d\tau} = \Omega_n - k_n z \tag{8c}$$

$$\frac{dy}{d\tau} = \sin(x-\Psi) + (g-2)y - \frac{g-1}{g} z \tag{8d}$$

$$\frac{dz}{d\tau} = gy - z \tag{8e}$$

Where $\Omega_n = \Omega T$, $k_n = kT$, $f_n = \frac{\omega_m}{a} = \omega_m T$ are the normalised detuning, normalized loop gain and nomalized frequency of modulating signal. The dynamics of third order PLL with CW input is described by the last three equations with Ψ=0.

Using these generalized system-differential equations with =0, the stability criterion can be determined by techniques of nonlinear control theory. It is quite apparent that the steady state values of the variables x, y and z are $\sin^{-1}\left(\frac{\Omega_n}{k_n g}\right)$, $\frac{\Omega_n}{k_n g}$, $\frac{\Omega_n}{k_n}$ respectively. Evaluating the determinant of the system Jacobian at these steady state points, one can derive the characteristic equation for the system as given in (9).

$$\lambda^3 + (3-g)\lambda^2 + \lambda - \sqrt{k_n^2 g^2 - \Omega_n^2} = 0 \tag{9}$$

By applying the Routh's Array technique, we obtain the conditions for stability as:

$$g < 3 \tag{10a}$$

$$\sqrt{k_n^2 g^2 - \Omega_n^2} > 0 \tag{10b}$$

$$(3-g) > \sqrt{k_n^2 g^2 - \Omega_n^2} \tag{10c}$$

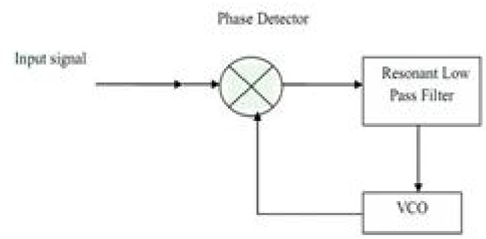

Figure 1. Block diagram of a conventional PLL

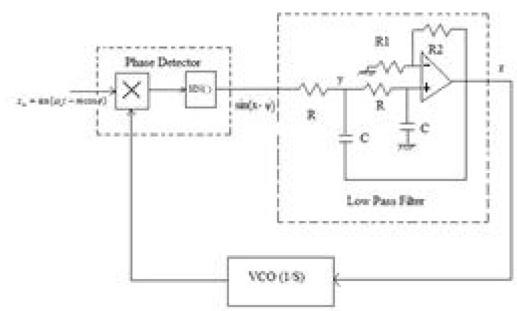

Figure2. State-space model of the third order PLL with FM signal showing the state variables φ,Ψ,x, y and z

III. Numerical Simulation And Results

The dynamics of the system would be completely understood by obtaining the solutions of the system equations (8a) to (8e). However they are nonlinear differential equations and hence closed form solutions are difficult to obtain for all parameter values, if not impossible. Hence numerical solutions of these equations are obtained by Fourth order Runge-Kutta technique and the system dynamics is examined for a range of values of the design parameters like *m*, g, Ω_n and k_n. Varying the parameters , (Ω_n and k_n being fixed) respectively the values of the state variables x,y and z are calculated. z(τ) being the VCO control voltage, the nature of VCO-output signal could be easily obtained from the solutions.

First we simulate the equations for simple CW input with just increasing the gain. With the increase of gain parameter, the system undergoes into the chaotic regime at g=2.5 through a period doubling sequence. Fig. 3 compares the

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phase-plane diagram of simulation results with that of the experimental results in this regard. Fig. 4 summarized the dynamics of PLL with CW input through a bifurcation diagram with gain as control parameter.

In the second part we study the dynamics of the system with FM input. We plot the phase-plane diagram with state variables y(τ) and z(τ) for increasing values of *m*, modulation index of FM signal. Observation shows the PLL transits from a steady state condition to a limit cycle state and then goes to chaotic state. (Fig. 5,6 and 7).

Here we verified the occurrence of chaos for suitable system-parameter values, using standard nonlinear dynamical measures like Lyapunov Exponents (λ_i) in both cases. The methodology adopted by Wolf et al. 1985[6] has been used to calculate the Lyapunov exponents from the system differential equations.

Table-1 gives the value of Lyapunov exponent for third order PLL with CW input and Table 2 is that of second case with FM input for different values of modulation index *m* and in tune condition.

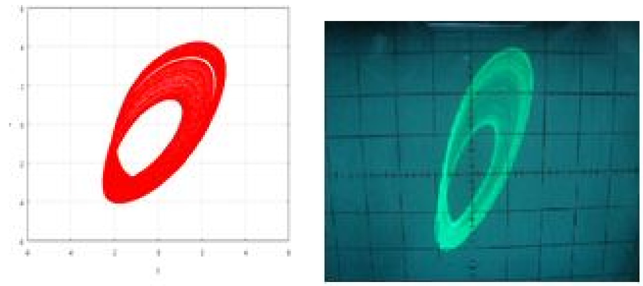
In Table-1 above 2.5 we get high value of positive lyapunov exponent. For FM input signal the results as shown in Table-2 indicate that when $m > 1.99$, we have high value of λ^+, which is indicative of the chaotic dynamics occurring in the system. We also calculate the maximum lyapunov exponent [MLE] from the time series data using users friendly software of time series analysis [7-9] and get $\lambda_i = 2.39$ for CW input at g=2.5, λ_i = -0.17 and 0.28 for m=1.9 and 1.99 when the system is in tune. In off tune carrier condition λ_i = -0.01 and 0.91 for m=1.26 and 1.34 which supports the chaotic oscillation.

TABLE I. LYAPUNOV EXPONENTS FOR DIFFERENT VALUES OF GAIN (FOR $K_N = 1$ AND $\Omega_n = 0.5$)

Gain parameter g	λ_1	λ_2	λ_3
1.3	-0.073115	-0.073462	-1.5534
1.5	-0.013655	-0.013736	-1.4726
1.7	0.0010543	-0.16503	-1.136
2.1	0.0040214	-0.027036	-0.87699
2.5	0.095223	-0.00011108	-0.59511
2.9	0.3387	0.0012514	-0.43995

TABLE II: LYAPUNOV EXPONENTS (Λ_i'S) FOR DIFFERENT VALUES OF M (FOR $g = 1.728, f_n = 0.904, \Omega_n = 0, k_n = 0.6511$)

m	λ_1	λ_2	λ_3	λ_4	λ_5
0	0	0.0001	-0.0428	-0.0428	-1.7482
0.1	0.0022	-0.0021	-0.0777	-0.7783	-1.6784
1.89	0.0025	-0.0021	-0.0867	-0.5605	-1.1869
1.97	0.0024	-0.0016	-0.0935	-0.6146	-1.1264
1.99	0.0147	0.0000	-0.0074	-0.6724	-1.1687
2.1	0.1060	0.0024	-0.0019	-0.7322	-1.2081



(a) (b)

Figure 3. Comparison of chaos in Third order PLL with CW input (a) Numerical result (b) Experimental result.

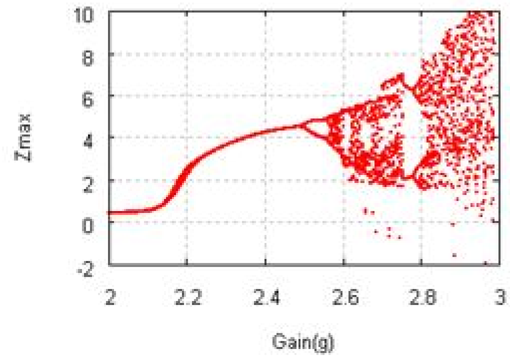
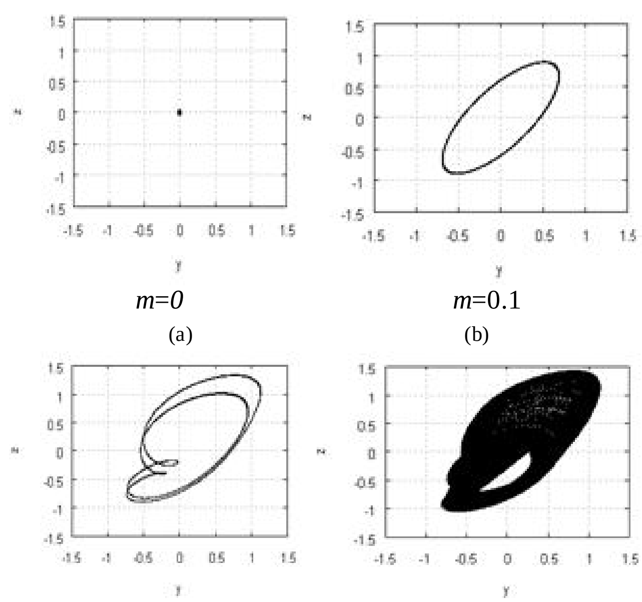


Figure 4. Bifurcation diagram of z with gain as the control parameter and k_n= 0.42,Ω_n= 0.22.

Intune



m=0 (a)

m=0.1 (b)

m=1.89 (c)

m=2.1 (d)

Figure 5. Phase plane plot of PLL operation, with , $g = 1.728, f_n = 0.904, \Omega_n = 0, k_n = 0.6511,$ and the modulation index *m* for (a) 0, (b) 0.1, (c) 1.89, (d) 2.1.

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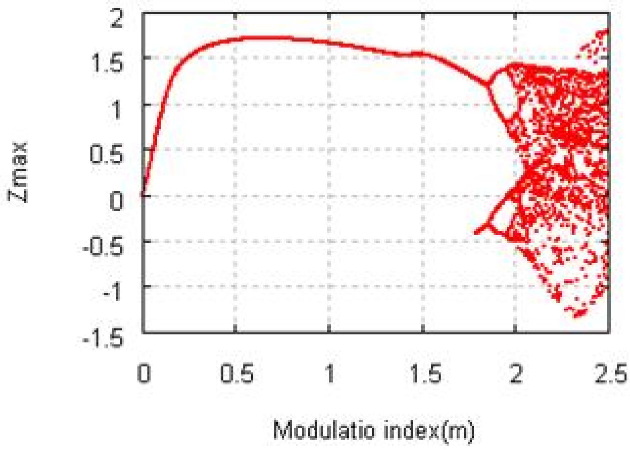
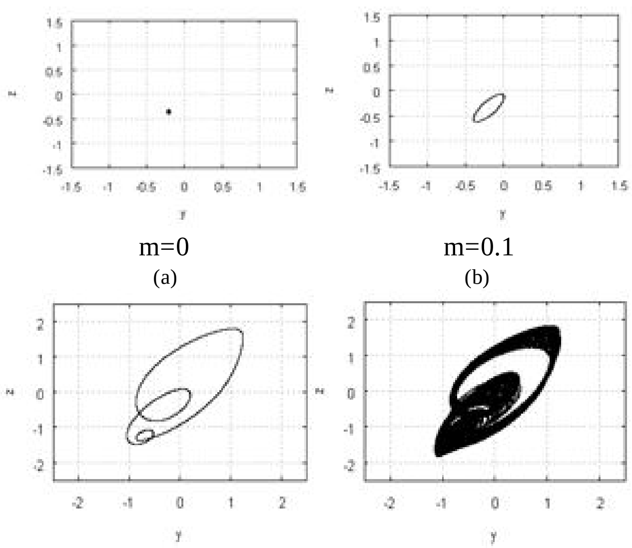


Figure 6. Bifurcation diagram of z with modulation index as the control parameter and k_n= 0.65,Ω_n= 0, g =1.728

Off Tune



m=0 (a) m=0.1 (b)

m=1.26 (c) m=1.34 (d)

Figure 7. Phase plane plot of PLL operation, with ,

$$g = 1.728, f_n = 0.769, \Omega_n = 0.2261, k_n = 0$$

and the modulation index *m* for (a) 0, (b) 0.1, (c) 1.26, (d) 1.34

IV. Experimental Studies

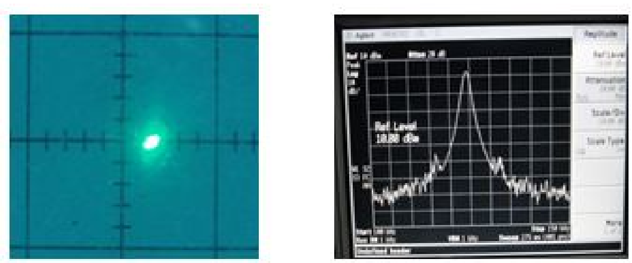
A prototype hardware circuit for a third order PLL with a resonant second order filter has been designed using off-the-shelf ICs and passive circuit components. The integrated circuit chip AD532 has been used to realize the multiplier-type phase detector used in the PLL structures. The active low-pass filters have been constructed using µA741 OP-AMPs and necessary passive circuit components. Aplab-make signal generator with FM modulation facility using external modulating signal is used as the VCO of the PLL. In the first case, the central frequency of the VCO is taken as 125 kHz and 127 kHz is that of CW input with 2 volt amplitude. The carrier frequency of the FM signal is also taken as 125 kHz for in tune. The spectrum of the VCO output is observed using Agilent-make spectrum analyzer. We observe a broad-band output spectrum in case of chaotic oscillation. The gain parameter of the loop filter is varied by changing the feed-back resistance at the filter circuit. Filter gain is 1.728 in the second case, i.e. within the stable region. The frequency of the modulating signal is 8 kHz in case of in tune and that is 6.8 kHz in case of off tune.

Fig. 3 shows the chaos obtained in third order PLL with CW input. The experimental results with FM input of some specific cases are shown in Fig. 8 to 10. Fig. 8 represents the steady state, as is evident from single component at the output spectrum of the VCO. In Fig. 8, the value of *m* is 0. At this condition the PLL is operated by simple sinusoidal signal or CW signal. For *m*=11.94, the loop is in chaotic state of oscillation which is clear from Fig. 9 as its spectrum is a broad one. Fig. 10 shows the chaos when the carrier signal frequency is 127 kHz i.e. in off tune. Experiments have been carried out with several other values of the loop-design parameters and obtained results are in close agreement with the analytically predicted results obtained via numerical solutions of system-equations.

V. Discussions

In the present paper, we have described the response of a class of third-order PLLs with a CW input as well as an FM input signal. The loop filter is a resonant type second order one. The chaotic oscillations are produced in the loop for a suitable set of loop-design parameters like loop filter-gain and time-constant, amplitudes of the input reference signal and the VCO signal, the phase detector gain, VCO sensitivity etc. Chaos has been observed for both CW and FM inputs. It is important to note that the dynamics of an FM demodulator may occasionally become chaotic if the signal parameters (like amplitude, modulation index etc) cross a finite limit depending on the values of loop design parameters. The results of numerical simulation are in close agreement with those of the hardware experiment.

The present study is useful in several respects. It shows that a properly designed third order PLL can be used as a RF chaos generator. Thus a chaotic modulator can be designed with it. Secondly, it indicates that design imperfections can make the output of a PLL demodulator can become chaotic. Hence a new degree of freedom (namely, modulation index) can be had for a controlled chaos generator with FM input signal.



(a) (b)

Figure 8. Amplitude of modulating signal V_{max}=0; (a) Phase plane plot and (b) Spectrum of VCO output voltage with Modulating signal Frequency=8kHz,Frequency of carrier signal=125kHz, Gain of PLL-1.728.

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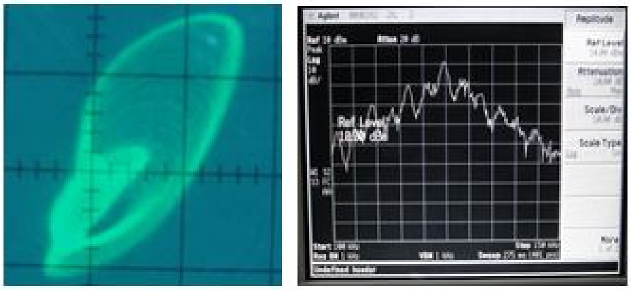
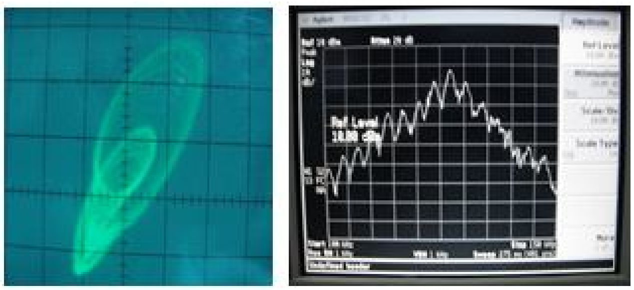


(a) (b)

Figure 9. Amplitude of modulating signal V_{max}=3.98; (a) Phase plane plot and (b) Spectrum of VCO output voltage with Modulating signal Frequency=8kHz,Frequency of carrier signal=125kHz, Gain of PLL-1.728.



(a) (b)

Figure 10. Amplitude of modulating signal V_{max}=2.18; (a) Phase plane plot and (b) Spectrum of VCO output voltage with Modulating signal Frequency=6.8kHz,Frequency of carrier signal=127kHz, Gain of PLL-1.728.

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